XXI. Tracking and Orbit Determination

MISSION ANALYSIS DIVISION

A. Spectral Factorization in Discrete Systems,

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1. Introduction

The spectral factorization is the basic problem arising in the Wiener–Shannon filtering theory. Usually the characteristic equation is solved and the roots having negative real parts are picked up in order to formulate a realizable filter in the continuous system. If the system is discrete, the characteristic roots lying inside the unit circle around the origin on the complex plane are selected for this purpose.

In contrast to these static filters, dynamic filters such as Kalman filters have been introduced in recent years which can be extensively applied to estimation problems in real time.

The sequential (Kalman) filter is asymptotically stable if the system is uniformly and completely observable and controllable (the definitions of these terms are contained in Ref. 1). Particularly when the system is time-invariant, the error covariance $P(t)$ converges to a constant matrix which is identical to the one obtained by the spectral factorization of the Wiener–Shannon theory.

According to Potter and Anderson (Refs. 2 and 3), this spectral factorization problem can be solved for continuous systems by finding the eigenvectors of a matrix which is constructed by the coefficient matrices of the Riccati equation (covariance equation).

In this article, the counterparts of their theorems in discrete systems are derived and the technique is demonstrated by an example.

2. Mathematical Results

Let the process equation of parameter $x\,(n \times 1)$ and equation of data $y\,(m \times 1)$ of a discrete system be

$$x(k+1) = \Phi(k)\,x(k) + G(k)\,w(k) \tag{1}$$

$$y(k) = H(k)\,x(k) + v(k) \tag{2}$$

where w and v are independent, white noises with zero mean. Hence the correlation of noise is

$$E[w(j)\,w'(k)] = Q(k)\,\delta_{jk} \tag{3}$$

$$E[v(j)\,v'(k)] = R(k)\,\delta_{jk} \tag{4}$$

$$E[w(j)\,v'(k)] = 0 \tag{5}$$

where δ_{jk} is the Kronecher delta and $E[\ ]$ indicates an ensemble average.

The error covariance $P(k)$ of the optimal estimate $x^*(k)$ is provided by the following Riccati equation (Ref. 1):

$$P(k+1) = \Phi(k)\{P(k) - P(k)H'(k) \times [H(k)P(k)H'(k) + R(k)]^{-1} \times H(k)P(k)\}\Phi'(k) + G(k)Q(k)G'(k) \tag{6}$$

with the initial condition

$$P(0) = E[x(0)x'(0)] \tag{7}$$

where the prime indicates the transpose.

In this article we are concerned with the solution of Eq. (6) when the system is time-invariant and the noises are stationary.

According to Kalman, Eq. (6) converges to a steady-state solution, provided the system is uniformly and completely observable and controllable. Assuming that this observability and controllability condition is satisfied, the solution of the following equation is sought in this article:

$$P = \Phi\{P - PH'[HPH' + R]^{-1}HP\}\Phi' + GQG' \tag{8}$$

As principal results of this article, three theorems are derived concerning the solution of Eq. (8). The proof of theorems may be referred to a JPL internal document.[1]

Theorem 1. *Let the matrix A $(2n \times 2n)$ be described by*

$$A = \left[\begin{array}{c|c} \Phi + GQG'\Phi'^{-1}H'R^{-1}H & GQG'\Phi'^{-1} \\ \hline \Phi'^{-1}H'R^{-1}H & \Phi'^{-1} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{9}$$

Also, let T be a matrix such that $T^{-1}AT$ reduces to a Jordan canonical form J. When T and J are similarly partitioned as A, the solution of Eq. (8) is given by

$$P = T_{11}T_{21}^{-1} \tag{10}$$

if T_{21} is non-singular.

[1]Nishimura, T., *Spectral Factorization in Discrete Systems*, May 12, 1970 (JPL internal document).

Theorem 2. *For an eigenvalue λ_i of A, there exists an eigenvalue $\lambda_j = 1/\lambda_i$. Namely, one-half of eigenvalues of A lie outside the unit circle on the complex plane while the other half lie inside the unit circle.*

Theorem 3. *Assuming all the eigenvalues of A are distinct, a real, symmetric, non-negative-definite solution P of Eq. (8) can be uniquely determined by arranging the sub-matrix*

$$\begin{bmatrix} T_{11} \\ \hline T_{21} \end{bmatrix}$$

such that it consists of eigenvectors of A corresponding to the eigenvalues which lie outside the unit circle.

3. Example

The solution provided by Theorem 3 is applied to the tracking problem of a spacecraft moving along a straight line by means of the ranging system at the ground station. The probe undergoes a small acceleration which is white with variable magnitude. The steady-state solution is sought in order to get the limiting accuracy of such tracking system.

The range data are taken every 1000 seconds and the data noise is taken as 10, 5, and 1 m, which is also assumed white. Let x_1 and x_2 be position and speed, respectively, of the spacecraft. Then

$$x_1(k+1) = x_1(k) + x_2(k) \tag{11}$$

$$x_2(k+1) = x_2(k) + w(k) \tag{12}$$

$$y(k) = x_1(k) + v(k) \tag{13}$$

Hence the coefficient matrices are

$$\Phi = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{14}$$

$$H = [1, 0] \tag{15}$$

$$R = \sigma_n^2 \tag{16}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_w^2 \end{bmatrix} \tag{17}$$

The limiting accuracies of this ranging system are demonstrated in Figs. 1 and 2. Figure 1 shows the uncertainty of position of the spacecraft versus the magnitude of acceleration while Fig. 2 does that of speed for different values of data noise.

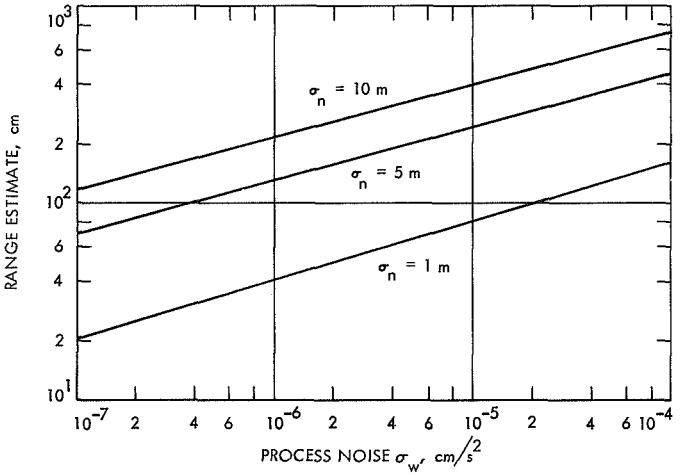

Fig. 1. Limiting accuracy of range estimate

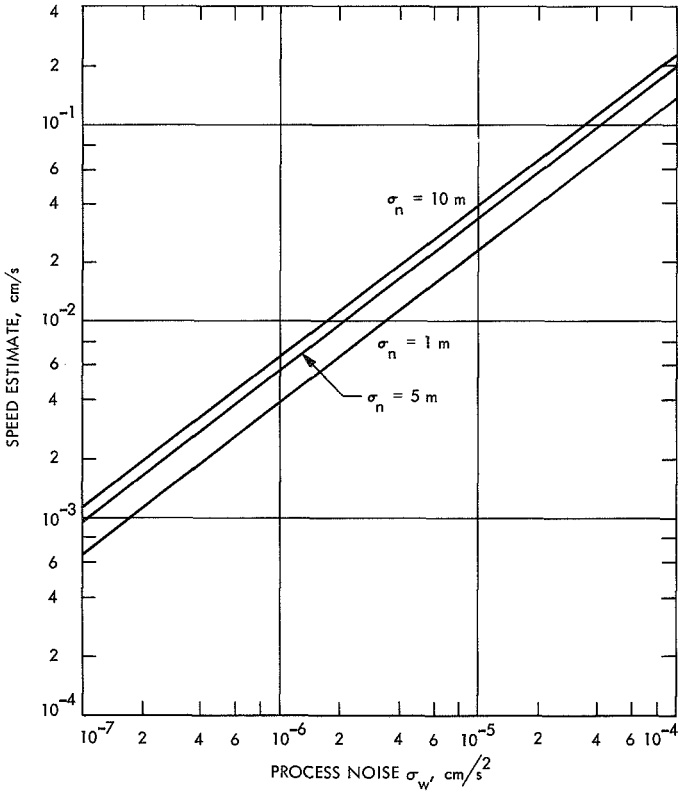

Fig. 2. Limiting accuracy of speed estimate

The range of process noise is such that $10^{-7} \sim 10^{-6}$ cm/s^2 is approximately the magnitude of solar disturbances and leakage in gas jets and $10^{-3} \sim 10^{-4}$ the magnitude of thrust fluctuations of solar electric spacecraft.

The discrete Riccati Eq. (8) for the system described by Eqs. (14–17) reduces to, for p_{11} (variance of estimate of x_1),

$$p_{11}^4 = \sigma_w^2 (p_{11} + \sigma_n^2)(p_{11} + 2\sigma_n^2)^2 \tag{18}$$

This is difficult to solve analytically. However, when $p_{11} << \sigma_n^2$, the solution is derived by

$$\sqrt{p_{11}} \approx 2^{1/4} \sigma_w^{1/4} \sigma_n^{3/4} \tag{19}$$

Therefore the standard deviation of the estimate of position is proportional to $\sigma_w^{1/4}$ of process noise and to $\sigma_n^{3/4}$ of data noise.

This agrees with the analysis in the continuous case for the same ranging system (Ref. 4). Since the inclination of straight lines in Fig. 1 is 1/4, it proves that $\sqrt{p_{11}}$ is proportional to the fourth root of standard deviation of process noise.

Similarly an approximate solution for p_{22} becomes

$$\sqrt{p_{22}} \approx 2^{1/4} \sigma_w^{3/4} \sigma_n^{1/4}$$

Namely, the standard deviation of estimate of speed is proportional to $\sigma_w^{3/4}$ of process noise and to $\sigma_n^{1/4}$ of data noise.

Again Fig. 2 shows that $\sqrt{p_{22}}$ is proportional to the three-fourth power of standard deviation of process noise.

References

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